Solving quadratic equations by completing the square
Perfect Square Trinomials

Before we solve equations by completing the square, we should be familiar with perfect square trinomials.

A perfect square trinomial is a trinomial like

\[ x^2 + 6x + 9 = (x + 3)(x + 3) = (x + 3)^2 \]
Let’s observe the pattern.

\[ x^2 + 6x + 9 \]

\[ x^2 = x \cdot x \]  The 1\text{st} term is a perfect square.

\[ 9 = 3 \cdot 3 \]  The last term is a perfect square.

\[ \frac{6x}{2} = 3x \]  Half of the middle term is the square root of the 1\text{st} term times the square root of the last term.
Perfect Square Trinomials

Other examples. \(4x^2 - 20x + 25 = (2x - 5)^2\)  
\(16x^2 + 8x + 1 = (4x + 1)^2\)

Notice: The last sign in the trinomial is always positive

The sign in the parentheses matches the sign of the middle term.
Completing the square

Completing the square is a way of artificially creating a perfect square trinomial.

Let’s start with an easy example:

\[ x^2 + 6x - 7 = 0 \]

Move the constant to the other side.

\[ x^2 + 6x = 7 \]

Leave a big old hole.

\[ x^2 + 6x + 9 = 7 + 9 \]

Fill the hole with \((\text{half of 6})^2\) or 9.

You have to add 9 to both sides.
Completing the square

Now let’s finish the problem.

Factoring our artfully constructed perfect square.

Taking the square root of each side.

Solving the 2 possible equations.
Medium hard problem

On to a harder problem.

\[ x^2 + 3x - 4 = 0 \]

\[ x^2 + 3x = 4 \]

Oops! The coefficient of x isn’t even.

We just go ahead and follow the pattern

\[ \left( \frac{3}{2} \right)^2 = \frac{9}{4} \]

Take half of 3 and square it.
Medium hard problem

Add \( \frac{9}{4} \) to each side:

\[
x^2 + 3x + \frac{9}{4} = 4 + \frac{9}{4}
\]

\[
(x + \frac{3}{2})^2 = \frac{16}{4} + \frac{9}{4}
\]

Use LCD to add fractions

\[
(x + \frac{3}{2})^2 = \frac{25}{4}
\]

Take square root of each side.

\[
\sqrt{(x + \frac{3}{2})^2} = \pm \sqrt{\frac{25}{4}}
\]

\[
x + \frac{3}{2} = \pm \frac{5}{2}
\]
Medium hard problem

Let’s finish up!

\[ x + \frac{3}{2} = \pm \frac{5}{2} \]
\[ x + \frac{3}{2} = + \frac{5}{2} \quad | \quad x + \frac{3}{2} = - \frac{5}{2} \]
\[ x + \frac{3}{2} - \frac{3}{2} = \frac{5}{2} - \frac{3}{2} \quad | \quad x + \frac{3}{2} - \frac{3}{2} = - \frac{5}{2} - \frac{3}{2} \]
\[ x = \frac{2}{2} \quad | \quad x = -\frac{8}{2} \]
\[ x = 1 \quad | \quad x = -4 \]
Hard problem

When you complete the square, the coefficient of the square term has to be 1.

\[ 2x^2 - x - 3 = 0 \]

What to do? What to do??

Divide everything by 2 – that’s what to do!
Hard problem

Now we have:

\[ x^2 - \frac{1}{2}x = \frac{3}{2} \]

Now we have to take half of the coefficient of \( x \) and square it.

\[ \frac{1}{2} \text{ of } -\frac{1}{2} = -\frac{1}{4} \]

\[ (-\frac{1}{4})^2 = \frac{1}{16} \]
Hard problem

We’re going to add \( \frac{1}{16} \) to each side:

\[
x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{3}{2} + \frac{1}{16}
\]

\[
= \frac{24}{16} + \frac{1}{16}
\]

\[
(x - \frac{1}{4})^2 = \frac{25}{16}
\]

Use LCD to add fractions.
Hard problem

Take the square root of each side:

\[ \sqrt{(x - \frac{1}{4})^2} = \pm \sqrt{\frac{25}{16}} \]

\[ x - \frac{1}{4} = \pm \frac{5}{4} \]
Hard problem

Add $\frac{1}{4}$ to each side and get 2 answers:

\[
x - \frac{1}{4} = \frac{5}{4} \quad \mid \quad x - \frac{1}{4} = -\frac{5}{4}
\]

\[
x - \frac{1}{4} + \frac{1}{4} = \frac{5}{4} + \frac{1}{4} \quad \mid \quad x - \frac{1}{4} + \frac{1}{4} = -\frac{5}{4} + \frac{1}{4}
\]

\[
x = \frac{6}{4} \quad \mid \quad x = -\frac{4}{4}
\]

\[
x = \frac{3}{2} \quad \mid \quad x = -1
\]
Completing the square

• The coefficient of the squared term must be 1
  • Divide if necessary
• Move the constant to the right of the =
• Take half of the coefficient of x and square it
• Add that to both sides of the =
• Use the LCD to add or subtract fractions